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A thermodynamic approach to the boundary layer flow system

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Abstract – A thermodynamic approach to the boundary layer flow system is used to investigate the control rule underlying the flow field. Application of Pontryagin's Maximum Principle from control theory shows minimum rate of entropy production to be the control rule for the flow field, in common with other dissipative processes. This result is used to investigate possible organised motion in the turbulent boundary layer and the generation of longitudinal vortices which is known to be an intrinsic characteristic of turbulence. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

One of the most important fundamentals remaining to be discovered in fluid dynamics is the mechanism or principle governing fluid motion. Principles governing motion are known in other branches of physics; for example Hamilton's Principle in analytical mechanics, where the governing equation and the principle are related through variational analysis. In the case, the physical meaning of Lagrange's function is quite clear: it shows the difference between the kinetic and potential energies underlying the mechanical motion. Much work has been carried out to find the variational principle in the field of fluid dynamics. However, to the best of the author's knowledge, Lagrange's function for the Navier–Stokes equations for steady incompressible flow has been shown to be non-existent by Brill [1], Millikan [2] and Gerber [3] (detailed discussions are reviewed by Serrin [4] and Finlayson [5]). The reason would appear to be partly due to the complex non-linear behaviour of fluid motion.

An alternative approach considered here is to apply the thermodynamic concept of irreversible processes (Prigogine [6], Grandsdorf and Prigogine [7]). Entropy increases in a dissipative field, while the rate of increase should be minimum in the case of the establishment of a stable field in an open system. As fluid flow is generally dissipative, the thermodynamic concept of irreversible processes is considered also to be applicable to fluid dynamics.

The purpose of this work is to investigate the applicability of the principle of minimization of entropy production rate as the control rule of fluid motion using a procedure from control theory developed by Pontryagin et al. [8], and to examine possible organised motion under this control rule. The boundary layer flow system is investigated as an interesting flow field for this purpose.

This paper is organised as follows. In order to apply a thermodynamic approach to the flow system, section 2 introduces the entropy equation in the general form for incompressible flow. Viscous dissipation is the only irreversible process in this case, and the dissipation function is re-written by using the Navier–Stokes equations

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and the continuity equation such that the entropy equation can be reduced to the diffusion equation of the entropy, with the entropy production term in the quadratic form of the vorticity in the flow.

Entropy change in the streamwise direction of the boundary layer flow is discussed using the equation, and integration of the cross-section normal to the main stream is performed for convenience so that the entropy equation is reduced to a one-dimensional linear differential equation with respect to the streamwise variation.

Section 3 investigates the control rule of the boundary layer flow along a flat plate using the previously derived equation. In this application, entropy is ever increasing in the downstream direction, reaching its maximum value at the downstream end boundary. For this physical reason, the Maximum Principle due to Pontryagin et al. [8] is applied instead of the variational method used in previous works (Serrin [4], Finlayson [5]) – the variation depends on the boundary, and while caution is needed when applying any variational methods, the condition at the end boundary is more relaxed when the Maximum Principle is applied.

In the Maximum Principle, the optimum process under given optimum conditions is considered to be the realisation of the maximum Hamiltonian along the path, and this is achieved by optimisation of the control function. In a flow system, the control function is the viscous dissipation which is determined by the flow field, and the problem in this case is to find the optimum condition which obtains the maximum Hamiltonian along the streamwise path. This optimum condition is considered to be the control principle underlying the boundary layer flow system.

Sections 2 and 3 consider the turbulence of the flow in the time-averaged form. Section 4 extends the control principle obtained to investigate the behaviour of the turbulence. The dissipation function in the original entropy equation is seen to reveal the entropy growth rate of the turbulence, and the possible patterns of turbulence satisfying the principle are obtained by the classical method of variation from the integration of the cross-section normal to the main flow. As the Navier–Stokes equations are not involved in this analysis, as opposed to section 2, the patterns derived appear to exhibit fundamental characteristics of turbulence, particularly longitudinal vortices, and the turbulent flow in the existing boundary layer is considered to be established in relation to the mean flow, Reynolds number, boundary conditions and so forth.

Comparison of the analytical results derived in this paper with experimental observations will validate the applicability of this approach to revealing the nature of the flow field.

2. Entropy equation

The general entropy equation for fluid motion per unit mass is shown in equation (1), where ρ and T denote respectively the density and absolute temperature of the fluid. As the present work concerns the dissipative process, viscous incompressible fluid is investigated so as to eliminate the change of fluid properties with temperature variation due to compressibility, in order to avoid complex interactions with the compressibility effect. ρ and T are both constants for the case under consideration. D/Dt and t denote respectively the substantial derivative and time, and Φ denotes the dissipation function due to viscosity.

$$\rho T \frac{Ds}{Dt} = \Phi. \tag{1}$$

Boundary layer flow is generally known to be continuous, and entropy change in the boundary layer is also therefore assumed to be analytically regular. For a boundary layer with viscous dissipation, s is taken to be the entropy increase over that of the external isentropic flow. Using the Navier–Stokes equation and the continuity equation, Φ can be rewritten as the sum of the diffusion term of the static pressure p and the quadratic term of the three-dimensional vorticity vector $\boldsymbol{\omega}$, where μ and ν denote respectively the viscosity and kinematic viscosity of the fluid (Aihara [9,10])

$$\rho T\frac{Ds}{Dt} = -2\nu\nabla^2 p + \mu\boldsymbol{\omega}^2, \tag{2}$$

where ∇^2 is the Laplacian operator.

The thermodynamic relation $\mathrm{d}h = T\,\mathrm{d}s + \mathrm{d}p/\rho$ is used to eliminate p from equation (2), the static enthalpy h being constant in this case. Equation (2) is then made dimensionless using the entropy of the external flow s_0 as the reference entropy, the boundary layer thickness δ as the reference length, the main flow velocity U as the reference velocity, and δ/U as the reference time, thus:

$$\frac{\partial s}{\partial t} + \nabla\cdot\mathbf{v}s = \frac{2\nu}{U\delta}\nabla^2 s + \frac{\nu}{Us}\frac{U^2}{Ts_0}\boldsymbol{\omega}^2, \tag{3}$$

where $\mathbf{v}$ and ∇ denote respectively the non-dimensional velocity vector and gradient operator.

In equation (3), s and $\mathbf{v}$ are generally composed of the time-means $\overline{s}$ and $\overline{\mathbf{v}}$ and the fluctuations s' and $\mathbf{v}'$. Applying Bousinesq's assumption $\overline{\mathbf{v}'s'} = -\varepsilon\nabla\overline{s}$, the time-averaged form of equation (3) is given by

$$\nabla\cdot\overline{\mathbf{v}}\overline{s} = \nabla\cdot\left(\frac{2\nu+\varepsilon}{U\delta}\right)\nabla\overline{s} + \frac{\nu}{U\delta}\frac{U^2}{Ts_0}\left(\overline{\boldsymbol{\omega}}^2 + \overline{\boldsymbol{\omega}'^2}\right), \tag{4}$$

where ε is the turbulent diffusivity of s. As the entropy tends to increase and the flux unifies the entropy in the field, it seems reasonable to assume that the flux is directed from the higher entropy region to the lower.

The case of the boundary layer along a flat plate will now be investigated as an example of a typical boundary layer flow. Here, the parallel flow approximation, $\delta = \overline{\delta} = \text{constant}$, is used for analytical convenience. This is considered applicable to a high Reynolds number flow except for in rapidly growing regions of the boundary layer, such as near the leading edge or in the transition region; that is, where the boundary layer thickness can be considered to vary slowly compared to the streamwise entropy growth, it appears that the latter can be analysed approximately under the assumption of parallel flow. This approximation has been applied successfully in the linear stability analysis of laminar boundary layers, for instance.

As the integral method is used in the following analysis, trial functions for $\overline{\mathbf{v}}$ and $\overline{s}$ are introduced, for instance by applying the Kármán–Pohlhausen procedure (Schlichting [11]). The boundary conditions to be satisfied by s are $\partial\overline{s}/\partial y = 0$ on the wall (that is, the increased entropy in the boundary layer is not transferred to the solid wall), and smooth approach to 0 at the external boundary. In this case, $\overline{\mathbf{v}}$ and $\overline{s}$ are written as

$$\begin{aligned}\overline{\mathbf{v}} &= \overline{\mathbf{v}}(y),\\ \overline{s}(x,y) &= x_1(x)\hat{s}(y),\end{aligned} \tag{5}$$

where x and y are the streamwise and normal coordinates respectively. The $x_1(x)$ term in the expression for $\overline{s}$ denotes that the streamwise entropy growth is dependent on the behaviour of the boundary layer flow. For another boundary condition, where $\overline{s}$ is given on the wall, an additional function of y is given on the right-hand side of $\overline{s}$ in equation (5). The latter, however, has no bearing on the following analysis since it is eliminated by differentiation with respect to x in equation (4).

Substituting equation (5) into equation (4) and integrating the result with respect to y from 0 to 1 and with respect to z from 0 to the spanwise characteristic scale of the turbulence or to the extent of the turbulence correlation l yields the following second-order differential form for x_1:

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}x^2} = k_1 \frac{\mathrm{d}x_1}{\mathrm{d}x} - k_2 m,$$
$$m \equiv \frac{1}{l} \int_0^1 \int_0^l \left(\overline{\boldsymbol{\omega}}^2 + \overline{\boldsymbol{\omega}'^2}\right) \mathrm{d}z\, \mathrm{d}y, \tag{6}$$

where

$$k_1 = \frac{Re}{2(1+\frac{\varepsilon}{2\nu})} \frac{\int_0^1 \overline{u}\, \hat{s}\, \mathrm{d}y}{\int_0^1 \hat{s}\, \mathrm{d}y}, \qquad k_2 = \frac{U^2}{2T s_0} \frac{1}{1+\frac{\varepsilon}{2\nu}} \frac{1}{\int_0^1 \hat{s}\, \mathrm{d}y}, \qquad Re = \frac{U\delta}{\nu};\ \text{Reynolds number},$$

and $\overline{u}(y)$ is the mean streamwise velocity distribution in the boundary layer.

3. Application of Pontryagin's Maximum Principle

Usually, Pontryagin's Maximum Principle is applied to obtain time-wise optimum control of a system from its characteristic state equations and a given optimum condition (Pontryagin et al. [8]). In the case under consideration, namely the boundary layer along a flat plate, the variable is the streamwise coordinate x. Before the present application is considered in more detail, the principle is summarised in a form convenient for the analysis as follows.

The state of the system is described by the state equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}x} = f_i\big(\mathbf{x}(x), \mathbf{m}(x), x\big) \quad (i = 1, 2, \ldots, n), \tag{7}$$

where $\mathbf{m}(x)$, generally the control vector, is the integration of viscous dissipation of the vorticity in the considered case. The initial condition of the state vector $\mathbf{x}(0)$ is given. State vector $\mathbf{x}(x)$ must be found which satisfies the given optimum condition

$$x_0 = \int_0^L f_0\big(\mathbf{x}(x), \mathbf{m}(x), x\big)\, \mathrm{d}x;\ \min \tag{8}$$

and the following necessary conditions:

(1) Hamiltonian $H = \sum_{i=1}^n p_i f_i$ is defined as the sum of the products of f_i and the auxiliary function $p_i(x)$, and x_i and p_i satisfy the canonical equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}x} = \frac{\partial H}{\partial p_i},$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}x} = -\frac{\partial H}{\partial x_i} \quad (i = 0, 1, 2, \ldots, n). \tag{9}$$

(2) H takes the maximum value H^* $(0 \leqslant x \leqslant L)$ along the optimum growth of the boundary layer, where L is the streamwise distance measured from the leading edge $(x = 0)$;

(3) p_0 is a negative constant, and p_i $(i = 1, 2, \ldots, n)$ are negative;

(4) when the end state $x_0(L)$ is not specified, $p_i(L) = 0$;

(5) when $f_i = f_i(\mathbf{x}, \mathbf{m})$, $H^* = 0$;

(6) when $f_i = \sum_{k=1}^{n} a_{ik} x_k(x) + g_i(\mathbf{m})$, $a_{ik} = \text{constant}$ $(i = 1, 2, \ldots, n)$, the sufficient condition of the Maximum Principle is verified.

In the present application, the state equations are derived from the entropy equation, equation (6), and are written in linear form as

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}x} &= x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}x} &= k_1 x_2 - k_2 m, \end{aligned} \tag{10}$$

where x_1 reveals the development of entropy as a function of x and x_2 reveals the spatial rate of change.

In the usual application of the Maximum Principle to control engineering, f_0 in equation (8) is given and the control vector $\mathbf{m}$ is adjusted to obtain maximum H along the integral path x, and the optimum process (H^*, $\mathbf{m}^*$) is achieved satisfying the optimum condition, equation (8). In the present case, as $\mathbf{m}$ is given from the vorticity of the flow field (equation (6)), the optimum process is only achieved by reasonable insight for f_0. The f_0 thus obtained is considered to give the control rule of the boundary layer flow system.

f_0 is generally obtained by solving the system of equations (9) and (10), where it is easily seen to be involved in the linear form. The uniqueness of the solution of the linear partial differential equation system for f_0 is an interesting subject. In the following analysis, however, f_0 is set more straightforwardly considering the present application of the thermodynamic concept of irreversible processes.

The right-hand sides of the expressions in equation (10) respectively show f_1 and f_2, and conditions (5) and (6) can be seen to be satisfied. For x_0, in order to investigate the applicability of the assumption of minimum entropy production rate [6,7], the integrand f_0 is taken to be x_2. The problem is now reduced to one of verifying the realisation of $H^* = 0$ at any value of x in the boundary layer with a physically reasonable m^*, the optimum value of m.

The Hamiltonian H and the canonical equations for $p_i(x)$ are written as

$$\begin{aligned} H &= p_0 x_2 + p_1 x_2 + p_2(k_1 x_2 - k_2 m), \\ \frac{\mathrm{d}p_0}{\mathrm{d}x} &= 0, \\ \frac{\mathrm{d}p_1}{\mathrm{d}x} &= 0, \\ \frac{\mathrm{d}p_2}{\mathrm{d}x} &= -p_0 - p_1 - k_1 p_2. \end{aligned} \tag{11}$$

Solving for $x_1 \sim p_2$ yields the following, where $d_1 \sim d_5$ are integration constants.

$$\begin{aligned} x_1 &= \int \mathrm{e}^{k_1 x} \left(\int \mathrm{e}^{-k_1 x} (-k_2 m) \, \mathrm{d}x \right) \mathrm{d}x + \frac{d_1}{k_1} \mathrm{e}^{k_1 x} + d_2, \\ x_2 &= \mathrm{e}^{k_1 x} \left(\int \mathrm{e}^{-k_1 x} (-k_2 m) \, \mathrm{d}x + d_1 \right), \\ p_0 &= d_3, \\ p_1 &= d_4, \\ p_2 &= \mathrm{e}^{-k_1 x} \left(\int \mathrm{e}^{k_1 x} (-p_0 - p_1) \, \mathrm{d}x + d_5 \right). \end{aligned} \tag{12}$$

From condition (4), integration constant d_4 is determined to be 0.

In the laminar boundary layer ($\varepsilon = 0$), m is given by Blasius's solution for the velocity distribution in the boundary layer (Schlichting [11]) and is a constant, m^*, under the approximation of parallel flow. The same is true for the fully-developed turbulent boundary layer where the streamwise change of the flow field, including the statistical distribution of the turbulence, is slow and so the same approximation can be adopted. Thus, in the turbulent boundary layer m is also regarded as a constant, m^*, although its value is much larger than in the laminar case. The values of k_1 and k_2 in the turbulent boundary layer also differ from those in the laminar boundary layer due to the change of diffusivity as can be seen from equation (6).

H^* is then determined to be as follows

$$H^* = \frac{d_3 k_2 m^*}{k_1} + d_1 d_5 k_1. \tag{13}$$

H^* is seen to be independent of x and constant along the direction of the flow. The integration constants d_1 and d_5 in equation (13) are now determined as follows.

Entropy continues to increase until the end state is reached, where its value will be at its peak in the system. Thus, $x_2(L) = 0$, and so d_1 is determined as

$$d_1 = -\frac{k_2 m^*}{k_1} \mathrm{e}^{-k_1 L}. \tag{14}$$

Since $p_2(L) = 0$ due to condition (4), d_5 is given by

$$d_5 = \frac{d_3}{k_1} \mathrm{e}^{k_1 L}. \tag{15}$$

Substitution of these results into equation (13) reveals

$$H^* = 0. \tag{16}$$

This result shows that the boundary layer flow system is formed under the optimum control rule of minimum entropy production rate derived from the thermodynamic principle of irreversible processes, where optimum control is performed by the dissipation m^* which takes different constant values in the laminar and turbulent regions. In other words, the boundary layer flow naturally observed is considered to be a typical dissipative system in the sense that the system is governed broadly by the concept of irreversible thermodynamics.

The relationship between the entropy increase and the behaviour of the boundary layer appears to be as follows. In the laminar boundary layer of the upstream region, the initial condition $x_1(0) = 0$ must be satisfied. Where the transition from laminar to turbulent flow does not take place until the end of the plate, L is taken to be the plate length D. Where transition occurs, L is regarded as being at the downstream end of the laminar region because entropy tends to reduce beyond this point and the flow system is not able to continue thermodynamically. The entropy growth in the laminar region is thus obtained to be as follows:

$$x_1 = \frac{k_2 m^* x}{k_1} - \frac{k_2}{k_1^2} m^* \mathrm{e}^{-k_1 L} \left(\mathrm{e}^{k_1 x} - 1\right). \tag{17}$$

After the transition, the boundary layer becomes turbulent and the value of L is always D. x_1 takes the same form as in the laminar case. The initial condition of the turbulent boundary layer will be given at its virtual origin, but is not always specified under the present approximation of parallel flow. From equation (17), the entropy production rate x_2 approaches 0 as x approaches L. Therefore, the application of the parallel flow approximation is seen to require a correction near the end state of the boundary layer flow.

This analysis shows that a common underlying control rule can be applied in both the laminar and turbulent regions of the boundary layer. In the transition region, however, the parallel flow approximation cannot be used and the concept of minimum rate of entropy production is not applicable; the transition is therefore beyond the power of the present analysis. The entropy growth in the boundary layer before and after transition is therefore discontinuous, entropy of course being higher after transition.

4. Formation of longitudinal vortices in the turbulent boundary layer as the result of Minimum Entropy Production

In the previous discussion, minimum growth rate of entropy was found to be the rule controlling the boundary layer flow system. For the turbulent boundary layer, the turbulence as well as the mean velocity distribution is treated as the statistical mean value in the previous sections. The underlying physical principle obtained for the mean flow is expected to be the same principle as that governing instantaneous turbulent motion. The analysis will now be extended to investigate possible organised motion in the turbulent boundary layer.

Returning to the original equation, equation (1), this is equivalent to the minimization of the integration of the dissipation function on the right-hand side with respect to y and z, as was done in equation (4). The velocity field is considered to be the superposition of the mean flow and turbulence as follows:

$$\begin{aligned} u &= \overline{u}(y) + u'(x, y, z, t), \\ v &= v'(x, y, z, t), \\ w &= w'(x, y, z, t), \end{aligned} \tag{18}$$

where the mean velocity field is assumed to follow the parallel flow approximation consistent with the analysis in the previous sections. The dissipation function is thus written as

$$\begin{aligned} \Phi = \mu\biggl\{ & 2\left(\left(\frac{\partial u'}{\partial x}\right)^2 + \left(\frac{\partial v'}{\partial y}\right)^2 + \left(\frac{\partial w'}{\partial z}\right)^2\right) + \left(\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial y}\right)^2 \\ & + \left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right)^2 + \left(\frac{\mathrm{d}\overline{u}}{\mathrm{d}y} + \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)^2 \biggr\}. \end{aligned} \tag{19}$$

By variational analysis, Lagrange's equation with respect to u' is obtained as

$$\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial(\frac{\partial u'}{\partial y})}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial(\frac{\partial u'}{\partial z})}\right) - \frac{\partial \Phi}{\partial u'} = 0 \tag{20}$$

and substitution for Φ then yields

$$\frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} - \frac{\partial^2 u'}{\partial x^2} + \frac{\mathrm{d}^2 \overline{u}}{\mathrm{d}y^2} = 0. \tag{21}$$

Likewise, from the Lagrange equations for v' and w', the following equations are derived.

$$\frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2} - \frac{\partial^2 u'}{\partial x \partial y} = 0, \tag{22}$$

$$\frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2} - \frac{\partial^2 u'}{\partial x \partial z} = 0. \tag{23}$$

From these equations, the process of establishment of the turbulent flow field can be seen; that is, u' is generated based on the mean flow in the boundary layer as shown by equation (21), and v' and w' are induced in response to u' as shown in equations (22) and (23) respectively. The analytical solutions of equation (21) are straightforward, and the turbulent flow corresponding to each solution will now be considered.

One of the homogeneous solutions for u' in equation (21) shows spatially periodic turbulences. For instance, u' is periodic in the yz-plane and the intensity changes streamwise, or u' is periodic in the xz-plane and the intensity distributes normal to the wall.

Another homogeneous solution for u' shows that the turbulence is confined in a cone having its axis aligned with the direction of the main flow and its apex at the origin of the disturbance. Experimental observations of the turbulence spot seem to show the similar patterns to this type of solution.

The inhomogeneous term of equation (21) reveals that the mean flow in the turbulent boundary layer plays the role of determining the original position and intensity of the various types of turbulence. From the linear form of equation (21), these turbulences are considered to occur randomly.

The elimination of u' from equations (22) and (23) results in the following Laplace equation in the yz-plane for the streamwise vorticity ω_x

$$\frac{\partial^2 \omega_x}{\partial y^2} + \frac{\partial^2 \omega_x}{\partial z^2} = 0, \quad \omega_x \equiv \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z}. \tag{24}$$

Equation (24) shows that in the formation of longitudinal vortices, the basic flow does not appear explicitly, and thus the generation of longitudinal vortices is considered to be intrinsic to the turbulent boundary layer.

Among the various possibilities which can result from equation (24), a pair of counter-rotating longitudinal vortices corresponding to the solution of the doublet is of interest in relation to the empirical observations of many investigations including the excellent review (Cantwell [12]) and other recent works. The generation of longitudinal vortices and the mechanism of energy supply to the turbulence through the breakdown of longitudinal vortices have been well investigated experimentally, while the mechanism of the generation of longitudinal vorticity due to turbulence has been investigated from the viewpoint of self-organisation of the turbulent boundary layer (Aihara [9]). The present result, that the generation of longitudinal vortices is intrinsic behaviour of the turbulent boundary layer, thus seems to be plausible considering empirical observations.

The distribution of the streamwise vorticity expressed by the doublet appears to show the state that the pair of concentrated vortices come closer to each other, which is often observed in the development of a pair of longitudinal vortices in the turbulent boundary layer, including Goertler vortices (Inagaki and Aihara [13]).

The results in this section are obtained by using the original entropy equation, equation (1). As opposed to the discussions in sections 2 and 3, the Navier–Stokes equations are not involved in the analysis. The resultant forms of turbulence are therefore considered to show the fundamental patterns of turbulence in the turbulent boundary layer. The multiplicity of patterns seems to correspond to entropy increase in the flow field.

The analysis developed in this section is restricted within the parallel flow approximation used for the mean flow of the boundary layers in sections 2 and 3. If one were to start from equation (20) without the restriction on the mean flow fields, admitting a priori the minimization of entropy production rate to be the general principle in a dissipative field including viscous flow, more general results would be obtained for a variety of viscous flow fields. However, the results for the boundary layer flow are found to be essentially the same as those derived this section, as far as the mean boundary layer development is not important.

5. Concluding remarks

A thermodynamic approach has been applied to clarify the control rule underlying the boundary layer flow system with the following conclusion.

The applicability of Prigogine's view of the thermodynamics of irreversible processes, that is, that the rate of entropy production is minimum in a stable, open system, was investigated in relation to the boundary layer along a flat plate and was found to be the control rule satisfying Pontryagin's Maximum Principle.

The result was extended to the turbulent boundary layer, and the generation of longitudinal vortices was found to be an intrinsic natural property associated with turbulence.

The thermodynamic analysis does not reveal the mechanism of the flow but nevertheless is considered to be useful in providing supplemental information to fluid dynamics.

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